Probabilistic Model Checking

Stefan Pranger

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Outline

- Motivation: New Models and New Properties

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- **Part I:** Modelling Formalism: The PRISM Language

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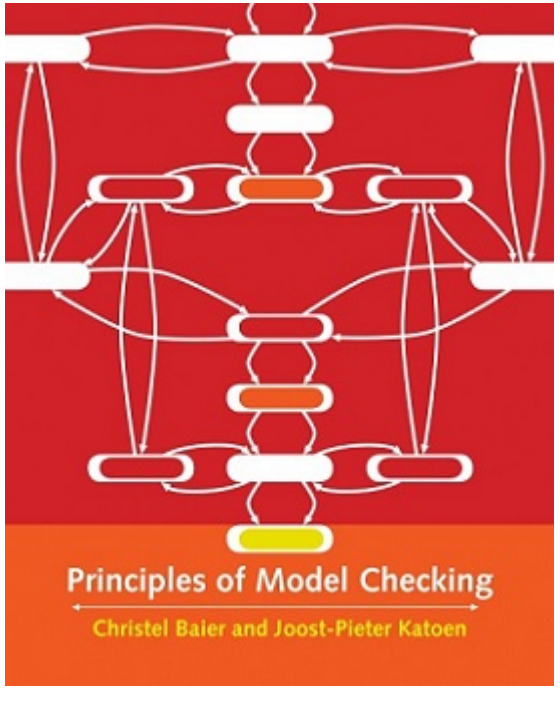
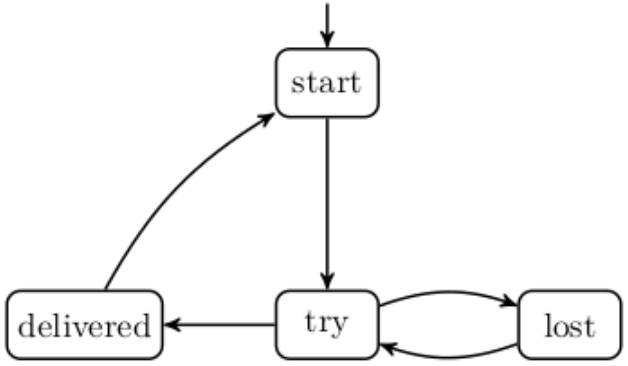
- **Part II:** Reachability Probabilities

Principles of Model Checking

We are switching to the second book!



Communication Protocol



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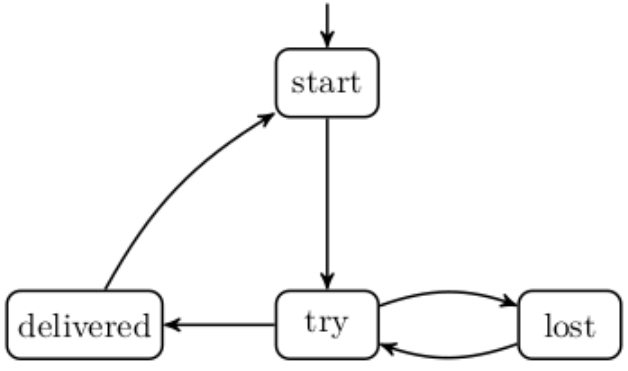

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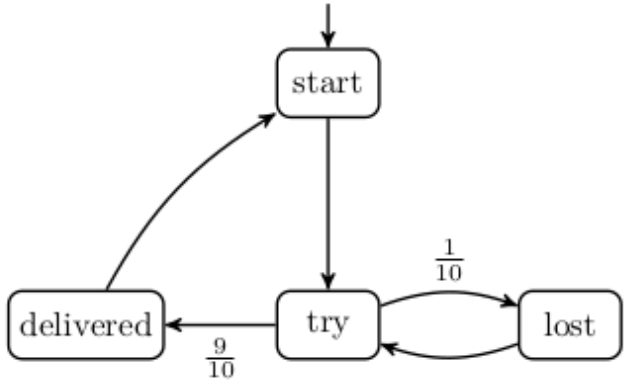

But $\mathcal{M}, start \models \exists \mathbf{G}\ \neg delivered$?

or $\mathcal{M}, start \models \forall \mathbf{F}\ delivered$?

- Does not capture probabilities! $\rightarrow$ We *need* new descriptions for properties . . .
- . . . and new models!

Markov Chains

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- AP set of atomic propositions and $L : S \to 2^{AP}$ a labelling function.

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- What is the probability to eventually send the message?
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- What is the probability to send 6 messages successfully and only failing a maximum amount of 15 times?

Part I

How do we describe models?

The PRISM Modelling Language

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  - **If** $processor\_one\_idle$ & $processor\_two\_idle$ **then** we can **decide** to use processor one or two.

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module foo
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  - We use it to describe the set of possible states and transitions between them.

The PRISM Modelling Language

- Formulas and Labels

```
formula num_tokens = q1+q2+q3+q+q5;
formula on_lava = x=WIDTH & y=HEIGHT;
formula crash = x1=x2 & y1=y2;
label "crashed" = crash;
label "violated" = crash | on_lava;
//[moveNorth] !crash & y<HEIGHT & ... -> ...;
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- Initial States

```
init
10 < x & x < 20
endinit
// or
init
true
endinit
```

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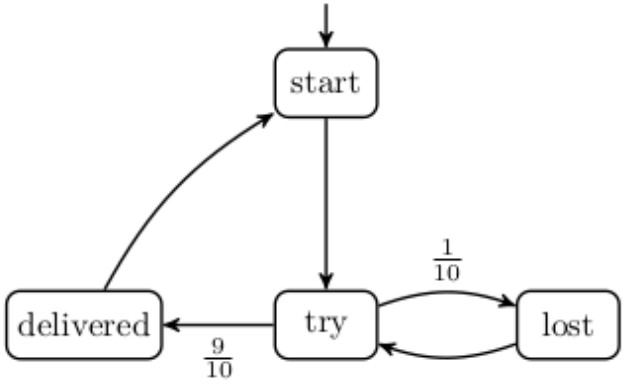


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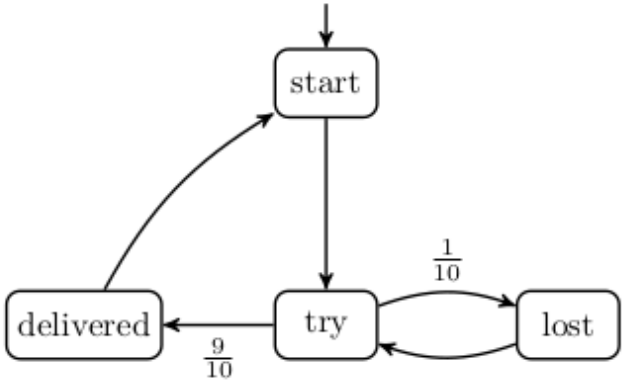


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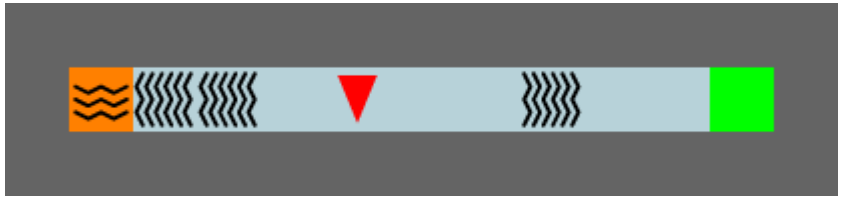
Live Coding!

Random Walk on a (Mini)Grid



* Orange means lava!
* Squiggly lines are cliffs which cannot be climbed
* Blue is slippery

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module 1D_robot

endmodule
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- Synchronization between modules
- Partially Observable Models
- Continuous-time Models
- Rewards
- etc.

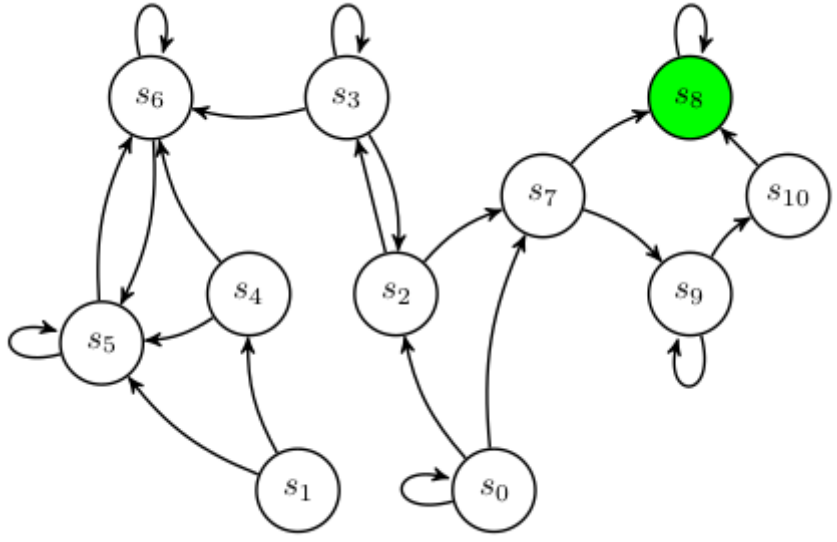
Part II

Reachability Probabilities

Probabilistic Reachability

- We start with objectives similar to the ones discussed at the beginning of the semester:

What is the probability that our system reaches its goal state?



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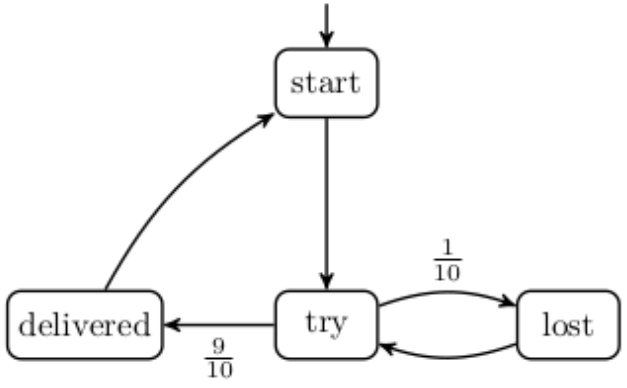

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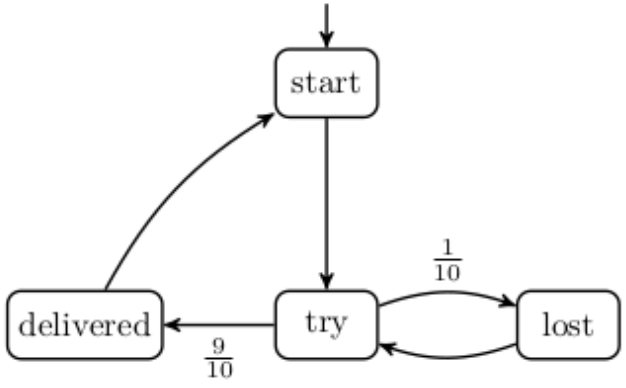

$$\mapsto \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & \frac{9}{10} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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  - What is the *probability* of my message to arrive after infinitely many tries?

$$Pr(\mathcal{M}, s \models \mathbf{F}\ \text{delivered})$$

Paths

- A *path* $\pi = s_0 s_1 s_2 \ldots \in S^\omega$, s.t. $\mathbb{P}(s_i, s_{i+1}) > 0, \forall i \geq 0$
- $Paths(\mathcal{M})$ is the set of all paths in $\mathcal{M}$ and
- $Paths_{fin}(\mathcal{M})$ is the set of all finite path fragments in $\mathcal{M}$.

Events and Paths

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- We capture such events via the *cylinder set*:

$$Cyl(\hat{\pi}_i) = \{\pi \in Paths(\mathcal{M}) \mid \hat{\pi}_i \in \text{pref}(\pi)\}$$

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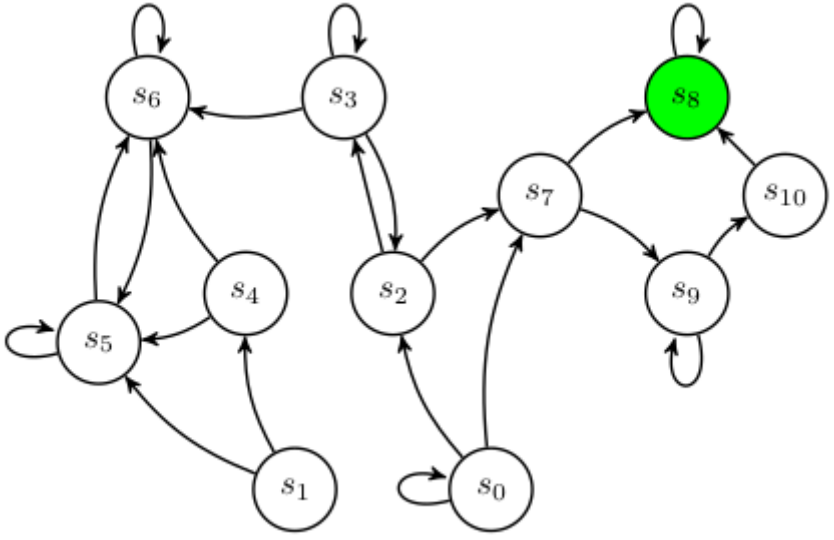
All $\hat{\pi} \in \Pi_{\mathbf{F}B}$ are pairwise disjoint, therefore:

$$Pr(\mathcal{M}, s_0 \models \mathbf{F}B) = \sum_{\hat{\pi} \in \Pi_{\mathbf{F}B}} Pr(Cyl(\hat{\pi}))$$

Computing $Pr(\mathcal{M}, s_0 \models \mathbf{F}B)$

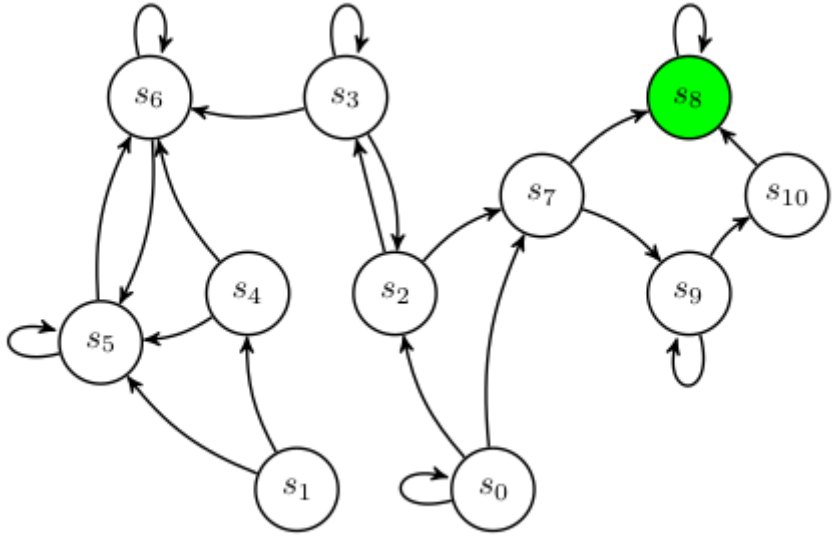


Computing $Pr(\mathcal{M}, s_0 \models \mathbf{F}B)$

2-step algorithm:

1) Identify three disjoint subsets of S:

- $S_{=1}$: The set of states with a probability of 1 to reach B.
- $S_{=0}$: The set of states with a probability of 0 to reach B.
- $S_{?}$: The set of states with a probability $\in (0, 1)$ to reach B.



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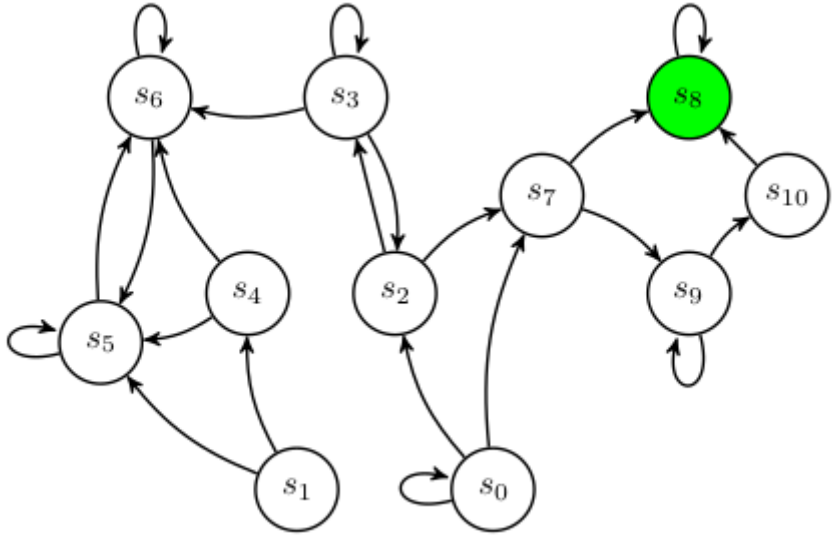
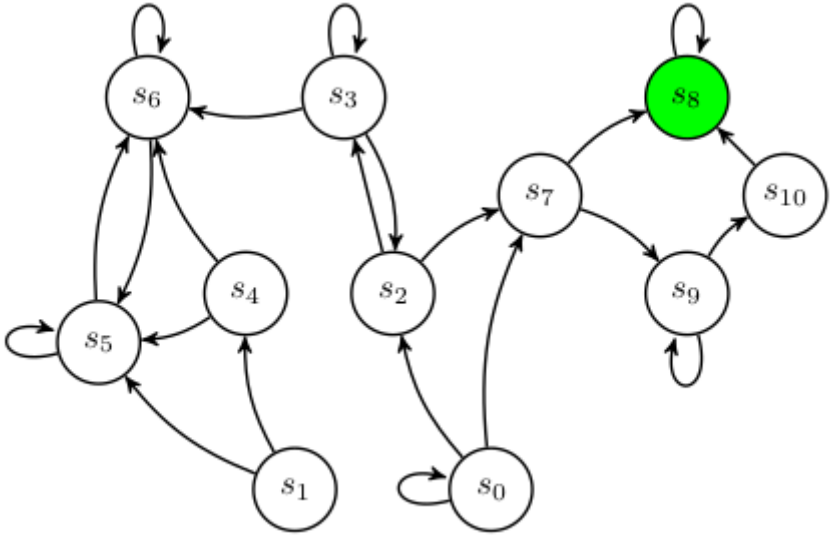
2) Compute the probabilities for all $s \in S_?$.



Computing $S_{=1}$ and $S_{=0}$

We can use graph-based analysis to compute these sets:



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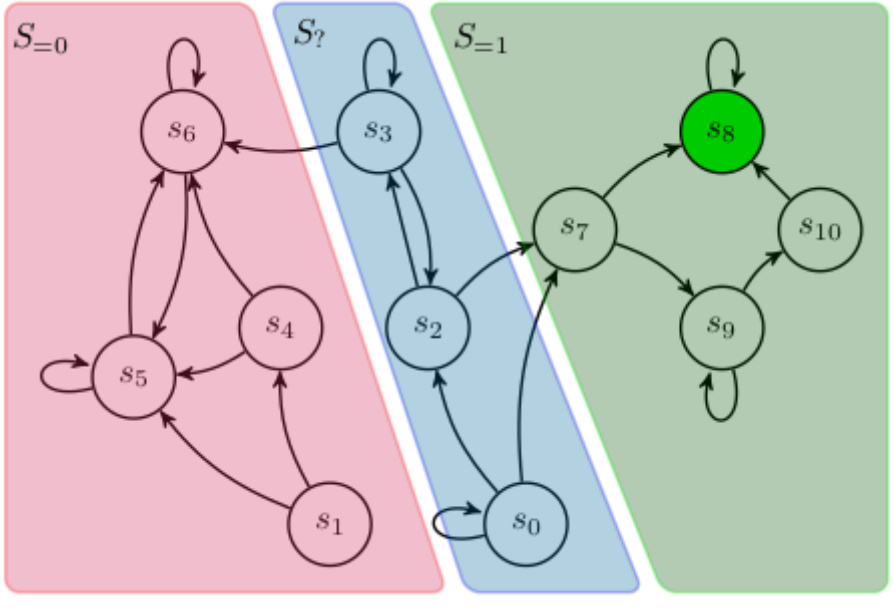

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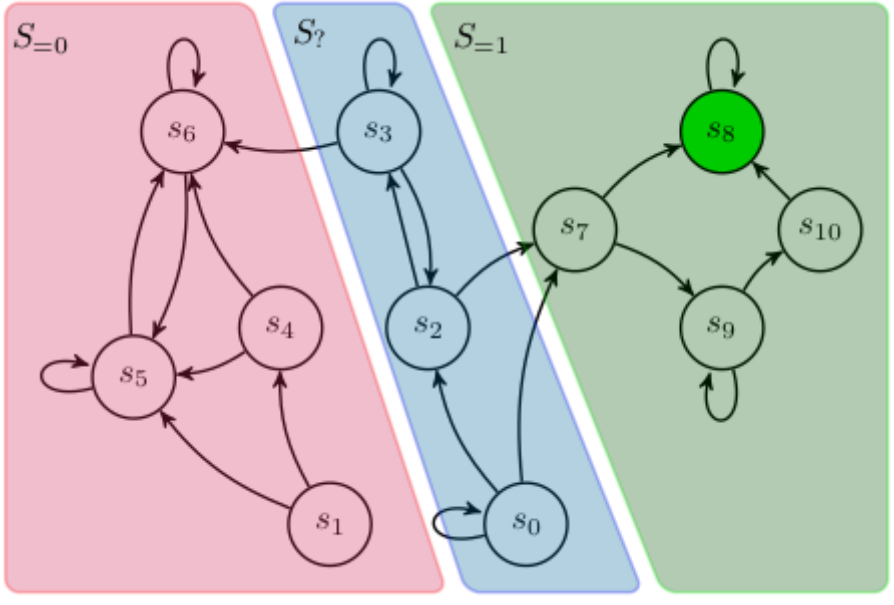

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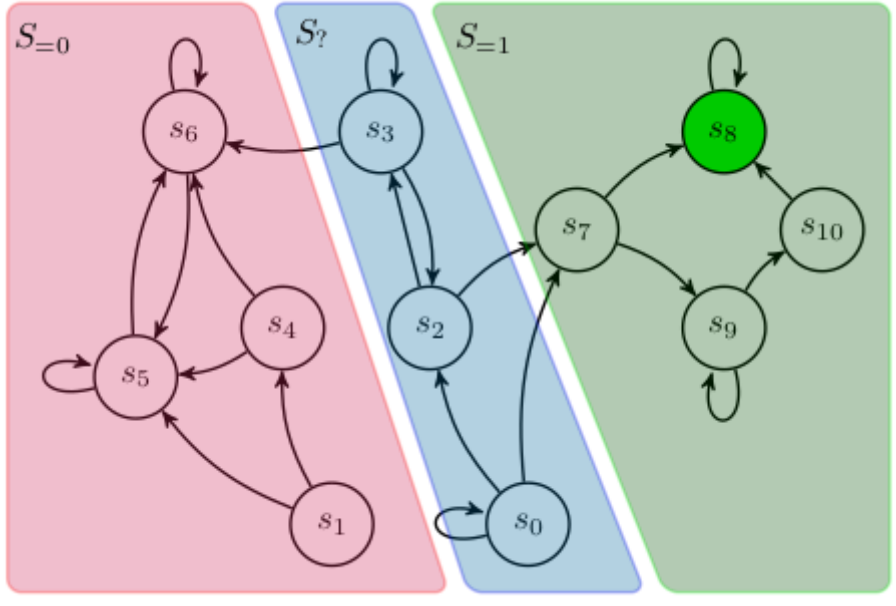
We compute the probability x_s of state s via:

- The probability to reach $S_{=1}$ in one step: $\sum_{u \in S_{=1}} \mathbb{P}(s,u)$
- and the probability to reach $S_{=1}$ via a path fragment $(s\ t\ \ldots\ u)$: $\sum_{t \in S_?} \mathbb{P}(s,t) \cdot x_t$
- Together

$$x_s = \sum_{t \in S_?} \mathbb{P}(s,t) \cdot x_t + \sum_{u \in S_{=1}} \mathbb{P}(s,u)$$



Computing $S_?$

Let us rewrite this into matrix notation:

- $A_? = (\mathbb{P}(s,t))_{s,t \in S_?}$
- $x = (x_s)_{s \in S_?}$
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$$x_s = \sum_{t \in S_?} \mathbb{P}(s,t) \cdot x_t + \sum_{u \in S_{=1}} \mathbb{P}(s,u) \rightsquigarrow x = A_? \cdot x + b = (I - A_?) \cdot x = b$$

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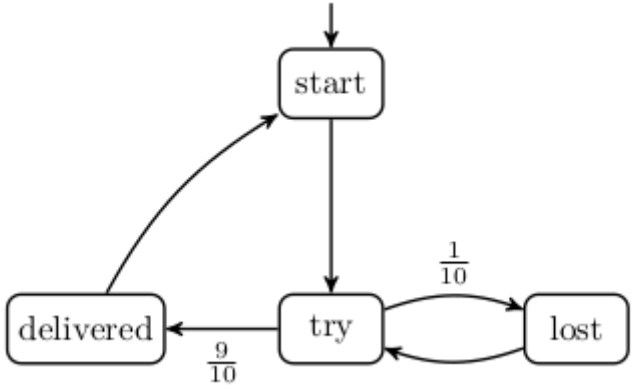
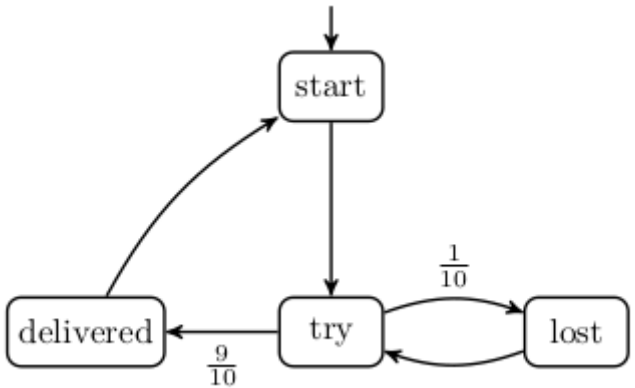


Communication Protocol



For this example we will not apply the graph-based analysis, i.e. we let $S_{=0} = \emptyset$ and $S_{=1} = \{\text{delivered}\}$.

$$\mathbf{A}_? = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ \frac{9}{10} \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{10} \\ 0 & -1 & 1 \end{bmatrix} \cdot \mathbf{x} = \begin{pmatrix} 0 \\ \frac{9}{10} \\ 0 \end{pmatrix} \rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Done

Transient State Probabilities

We will consider a slightly different algorithm:

$$\mathbf{A}^n = \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \cdot \dots \cdot \mathbf{A}$$

contains the probability to be in state t after n steps in entry $\mathbf{A}^n(s,t)$.

We call

$$\Theta_n^{\mathcal{M}}(t) = \sum_{s \in S} \mathbf{A}^n(s,t)$$

the *transient state probability* for state t.

Transient State Probabilities

Let's consider $(\Theta_n^{\mathcal{M}}(t))_{s \in S}$, the vector of transient state probabilities for the nth step.

We can compute $Pr(\mathcal{M}, s_0 \models \mathbf{F}^{\leq n} B)$ in a modified Markov chain:

$$\mathcal{M}_B = (S, s_0, \mathbb{P}_B, AP, L)$$

where:

- $\mathbb{P}_B(s,t) = \mathbb{P}(s,t)$ if $s \notin B$
- $\mathbb{P}_B(s,s) = 1$ if $s \in B$
- $\mathbb{P}_B(s,t) = 0$ if $s \in B$ and $t \notin B$

i.e. all $s \in B$ become sinks and B cannot be left anymore.

Transient State Probabilities

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We then have

$$Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B) = Pr(\mathcal{M}_B, s \models \mathbf{F}^{=n} B)$$

and therefore

$$Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B) = \sum_{t \in B} \Theta_n^{\mathcal{M}_B}(t)$$

Computing $Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B)$ via Transient State Probabilities

We have the following algorithm to compute $Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B)$:

- $\Theta_0^{\mathcal{M}}(t) = \mathbf{e}_i$, i.e. the unit vector with 1 at the ith position and 0 else.
- For $k = 0$ up to $n-1$: $\Theta_{k+1}^{\mathcal{M}}(t) = \mathbf{A} \cdot \Theta_k^{\mathcal{M}}(t)$
- $Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B) = \sum_{t \in B} \Theta_n^{\mathcal{M}_B}(t)$