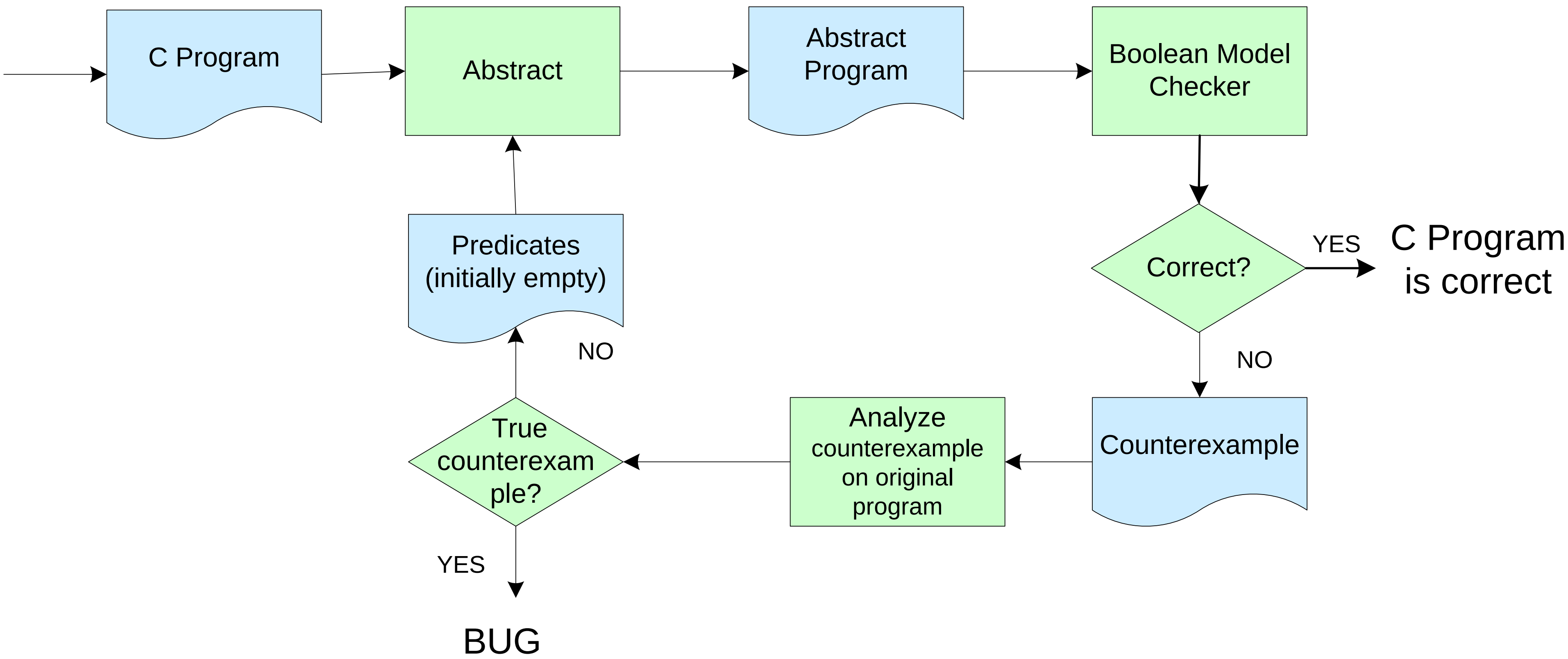
Abstraction

The Approach



Abstraction

- Represent complex program by simple program
  - original program is **concrete**, simple one is **abstract**
- Construction: if abstraction correct, then original correct
  - But: abstract program may fail even if the original is correct
  - We will look at *refinement* later
- Whenever we can not make a decision with certainty, we allow all possibilities

Predicate Abstraction

- Replace variables by predicates. E.g., instead of `x` have the predicates
  - `b, meaning {x>0},`
  - `c: {x<0},`
  - `d: {x==0}`
- or replace `x` and `y` by
  - `e: {x==y}, or by`
  - `f: {x<y}, or by`
  - `g: {2x - y < 0},`

Predicate Abstraction Examples

```
b: {x is odd}
assert(x!=38)
```

```
if(x==5) then S1 else S2 fi
```

Predicate Abstraction

Example: keep only the lowest bit of a number.

- `b: {x is odd}`
- `assert(x!=38)` becomes `assert(b)`
- `assert(b)` is stricter:
  - if `assert(x!=38)` fails then `assert(b)` fails
  - But not vice-versa

- `if(x==5) then S1 else S2 fi` becomes
  `if(b?*:F) then S1 else S2 fi`

(meaning: if b is true, try both branches, otherwise try only the else branch)

Construct abstract programs **one statement at a time**

Abstraction Example

For automatic abstraction, let's first check some basics.

Predicate: $b = \{x \leq y\}$

Abstract

`x := y?`

Computing Abstraction

`b = {x ≤ y}`

Use Hoare’s weakest precondition

```
{y ≤ y}
x := y
{x ≤ y}
```

Thus, $y \leq y$ before the statement iff $x \leq y$ after

`x := y` is abstracted to

`b = true`

Abstraction Example

For automatic abstraction, let's first check some basics.

Predicate: `b = {x ≤ y}`

Abstract

`y := y+1?`

b	b'

Computing Abstraction

Now for `y := y + 1`.

```
{x ≤ y + 1}
y := y + 1
{x ≤ y}
```

Thus, $x \leq y + 1$ before iff $x \leq y$ after.
In which cases can we guarantee $x \leq y+1$?

b	b'
{x ≤ y}	{x ≤ y+1}
T	T
F	*

Not enough information to decide whether $x \leq y+1$ before – approximate:

```
b = b ? T : *;
```

Program Abstraction – Line by Line!

```
b:{x < 0}.

x = -2;

x = x + 1;

assert(x<0);
```

Abstraction is *conservative*: bugs are preserved (but new bugs may occur).

Conservative Abstraction

Let us abstract `x` by **b**`:{x < 0}.`

We may loose some information Example:

```
x = -2;
x = x + 1;
assert(x<0);
```

is abstracted statement-by statement-to

```
b = true;
b = b ? * : false;
assert(b);
```

The abstraction is *conservative*: bugs are preserved (but new bugs may occur).

Two Predicates

Two predicates: `b={x ≤ y}` and `c={x=y+1}`. Statement: `y := y + 1`

b	c		b'	c'
x≤y	x=y+1			

`y := y + 1`

`y := y + 1`

Abstraction

Two Predicates

Two predicates: `b={x ≤ y}` and `c={x=y+1}`

preconditions:

```
{x ≤ y + 1}
y := y + 1
{x ≤ y}

{x = y + 2}
y := y + 1
{x = y+1}
```

`y:=y+1` is abstracted to

```
simultaneous
   b := b&&!c || !b&&c ? T : F
   c := b&&!c || !b&&c ? F : *
end
```

b	c		b’	c’
$x\leq y$	$x=y+1$		$x\leq y+1$	$x=y+2$
T	T	X		
T	F	$a \leq b$	T	F
F	T	$a=b+1$	T	F
F	F	$a>b+1$	F	*

In general, simultaneous assignments are needed for abstract statements

Abstraction of Conditional

* denotes nondeterministic value

b	
{x odd}	{x = 5}

Original Program

```
if(x == 5) then
  S1
else
  S2
fi
```

Abstract Program (b = {x odd})

Abstraction of Conditional

We use `*` to denote a nondeterministic value

`b`	
$\{x$ `odd}`	`{x = 5}`
T	*
F	F

Original Program

```
if(x == 5) then
  S1
else
  S2
fi
```

Abstract Program (`b = {x odd}`)

```
if(b?*:F) then
  S1
else
  S2
fi
```

Note:

- b=false is the same as x even, which implies x!=5.
- b=true means that x is odd, which means x may or may not be 5

Another Example

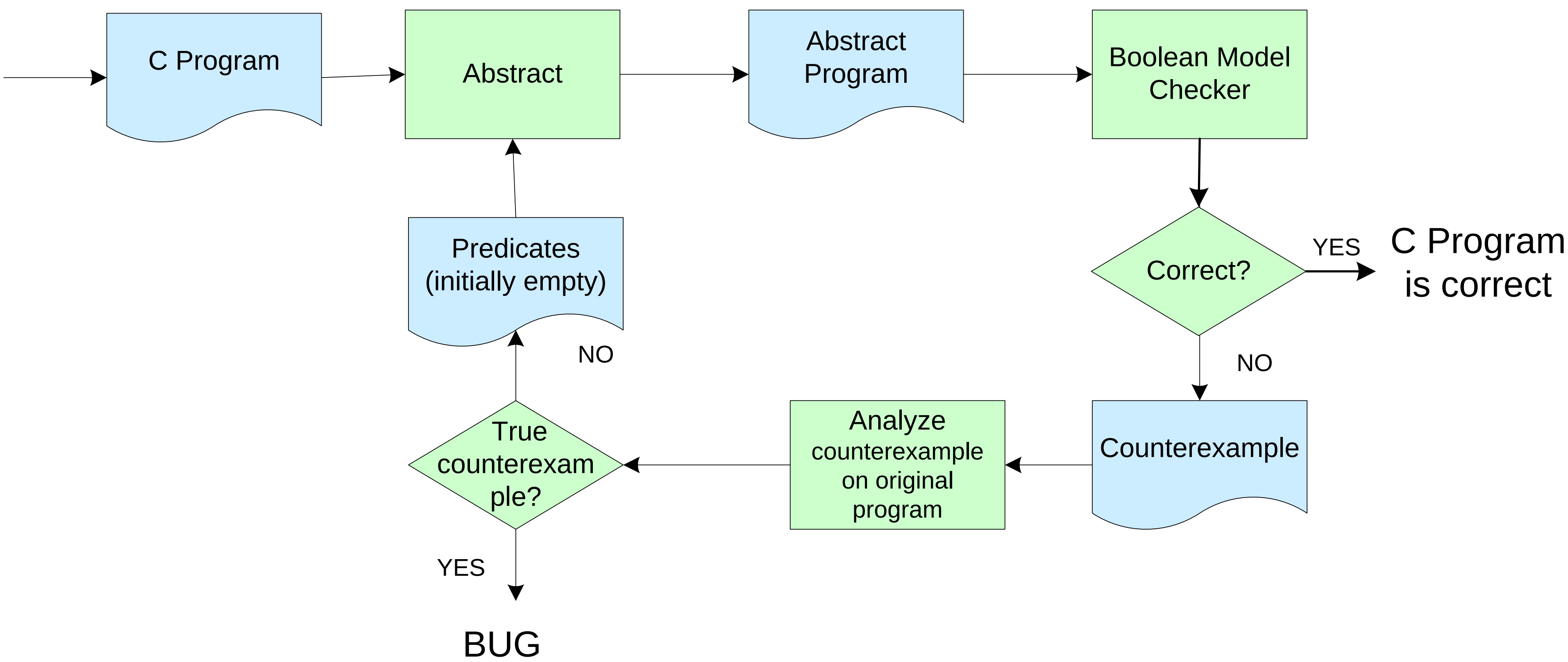
```
done = 0;
while(done == 0){
  if(x != 0)
    x--;
  else
    done++;
}
assert(x == 0);
```

Which predicates do we need?

Abstraction

- Tricky: find the proper abstraction!
  - You use the counterexamples, but how?
  - You can do it by hand
  - You can try to do it automatically
- Automatically finding the proper abstraction cannot always work. Why not?

The Approach



Precisely: assignment

Original: `x:= e`
Predicates p1,…,pn.

Suppose we have
```
{qi}
x := e;
{pi}
```

Let ai be the disjunction of assignments (rows in table) to p1…pn that imply qi.
Let bi be the disjunction of assignments (rows) to p1…pn that imply ¬qi.
The remaining assignments contain *

`x := e` is replaced by
```
simultaneous
  p1 = a1 ? T : b1 ? F : *
  …
  pn = an ? T : bn ? F : *
end simultaneous
```

example

Assignment: b := b+1
Predicates: `p1 = {a ≤ b}` and `p2 = {a=b+1}`

```
{a ≤ b + 1}    {a = b + 2}
b := b + 1     b := b + 1
{a ≤ b}        {a = b + 1}
```

```
Look at the table: row TT, TF, and FT have a T in
column a≤b and TT and FF have an F in that
column.  Therefore:
p1 ∨ p2             implies a ≤ b + 1
(p1∧p2)∨(¬p1∧¬p2) implies a > b + 1
(note: false implies anything)
```

For the 2nd predicate:
```
p1 ∧ p2 implies a = b+2
p1 ∨ ¬p2 implies a ≠ b+2
```

`b:=b+1` is abstracted to
```
simultaneous
 {a≤b}   := p1||p2 ? T : p1==p2 ? F : *
 {a=b+1} := p1&&p2 ? T : p1!=p2 ? F : *
end
```

p1	p2			
a≤b	a=b+1		a≤b+1	a=b+2
T	T	×	T/F	T/F
T	F	a≤b	T	F
F	T	a=b+1	T	F
F	F	a>b+1	F	*

(Cf. same example on an earlier slide)